on an airfoil at high incidence first appears when the stagnation point moves to ordinates exceeding about one-half the leading-edge radius

2) The lift on a truncated parabolic cylinder varies linearly with the ordinate of the stagnation point Extending to airfoils, an approximate relation between stagnation-point movement and change of lift, at any constant camber, is

$$(y_{\text{t g}})_2/r = (y_{\text{t g}})_1/r + 0\,02(c/r)(C_{L_1} - C_{L_2}) \qquad (10)$$

3) For an airfoil at its ideal angle of attack, the velocity gradient at the stagnation point is given by

$$dU/ds = U_\infty/r \qquad (11)$$

where s is distance along the wall In the limit, as $r \to \infty$, we know that $U_\infty \to \infty$ and $dU/ds \to a$ All possible ideal-incidence stagnation-point velocity gradients may then be summarized as $0 < r \to \infty$, $\infty > dU/ds \geq a$

These results provide a first step in the design of an airfoil from a known pressure distribution If $U = U(s)$ is known, then Eq (11) gives the airfoil leading-edge radius directly

Ablation of a Cylindrical Cavity in an Infinite Medium

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Nomenclature

a = initial radius of cylindrical cavity, in
c = specific heat Btu/(lb- R)
k = thermal conductivity, (Btu-in)/(ft²-sec R)
L = heat of sublimation Btu/lb
Q = heat flux, Btu/(ft²-sec)
r = radial coordinate in
s = radius of ablating heated surface, in
t = time, sec
T = temperature, °R
T_i = initial temperature, °R
T_M = melt or sublimation temperature, R
α = $(Qa)/[k(T_M - T_i)]$
β = $\rho L\kappa/[k(T_M - T_i)]$
δ = depth of thermal layer, in
κ = diffusivity, in ^{2}/sec
ρ = density of solid, lb/ft^3
θ = temperature, nondimensional, $(T - T_i)/(T_M - T_i)$
τ = nondimensional time $\kappa t/a^2$
ξ,ζ = nondimensional space coordinates, s/a, r/a
$(\dot{\ })$ = indicates differentiation with respect to τ
$(\ )'$ = indicates differentiation with respect to time

Introduction

ONE of the means by which engineers have coped with the problems of high thermal inputs to aerospace vehicles has been the use of an ablative heat shield Problems involving the transient temperature distribution in bodies undergoing phase changes and the study of the ablative process have thus received considerable attention in the literature In this note, a simple approximate heat-balance technique due to Goodman[1] is utilized to study the ablation of a cylindrical cavity in an infinite medium Further considerations, refinements, and applications of the technique have appeared in the literature [2–4] A comparison of the heat-balance technique with two other approximate methods was recently reported [5] Lardner[6] discussed the discrepancies between two approximate solutions and known exact solutions to the Stefan problem and the problem of a semi-infinite body under constant heat input

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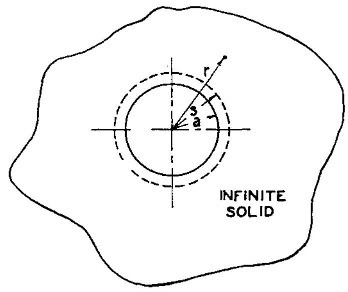

Fig 1 Notation for coordinate system

Statement of Problem and Assumptions

An infinite medium surrounds a cylindrical cavity The initial temperature T_i throughout the medium is constant, and a heat source in the cylindrical cavity radiates heat to the surface in an axisymmetric manner The initial radius of the cylindrical cavity is a (Fig 1) Time is reckoned from the instant that the heated surface reaches the melt or sublimation temperature T_M For $t > 0$, continued exposure of the heated surface to heat flux causes the material to ablate It is desired to obtain the time history of the ablating heated surface

The following assumptions are made in the analysis: 1) the heated surface remains at the melt or sublimation temperature T_M; 2) the melt or products of sublimation are immediately removed upon formation; 3) the heat flux Q remains constant; and 4) thermal properties of the solid are independent of temperature

The equation governing the axisymmetric flow of heat in the infinite region around the cylindrical cavity is most conveniently written in cylindrical coordinates:

$$k(1/r)(\partial/\partial r)[r(\partial T/\partial r)] = \rho c(\partial T/\partial t) \qquad (1)$$

Equation (1) is valid for $t > 0$, in the region $s(t) < r < \infty$

The initial and boundary conditions to be satisfied are

$$T(s,t) = T_M \qquad (2)$$

$$T(\infty,t) = T_i \qquad (3)$$

$$Q(s) = -k(\partial T/\partial r) + \rho L(\partial s/\partial t) \qquad (4)$$

$$s(0) = a \qquad s'(0) = 0 \qquad (5)$$

It is convenient to introduce the new variables

$$\theta = (T - T_i)/(T_M - T_i) \qquad \tau = \kappa t/a^2 \qquad (6)$$

$$\zeta = r/a \qquad \xi = s/a$$

The substitution of the variables defined in Eqs (6) into Eqs (1–5) yields the following form of the heat-conduction equation and the associated boundary and initial conditions:

$$(1/\zeta)(\partial/\partial\zeta)[\zeta(\partial\theta/\partial\zeta)] = \partial\theta/\partial\tau \qquad (7)$$

$$\theta(\xi,\tau) = 1 \qquad (8)$$

$$\theta(\infty,\tau) = 0 \qquad (9)$$

$$\alpha - \beta\dot{\xi} = -(\partial\theta/\partial\zeta)_\xi \qquad (10)$$

$$\xi(0) = 1 \qquad \dot{\xi}(0) = 0 \qquad (11)$$

where

$$\alpha = (Qa)/[k(T_M - T_i)] \qquad (12)$$

and

$$\beta = \rho L\kappa/[k(T_M - T_i)] \qquad (13)$$

Approximate Solution

The heat-conduction equation (7) is integrated over the volume of the solid

$$\int_{\xi}^{\infty} \frac{\partial}{\partial\zeta}\left[\zeta\left(\frac{\partial\theta}{\partial\zeta}\right)\right] d\zeta = \int_{\xi}^{\infty}\left[\frac{\partial(\theta\zeta)}{\partial\tau}\right] d\zeta \qquad (14)$$

Use is made of the Leibnitz rule and Eqs (8) and (10) to put (14) into the form

$$(\alpha - \beta\dot{\xi})\xi = \xi\dot{\xi} + \frac{d}{d\tau}\int_{\xi}^{\infty} (\theta\zeta)d\zeta \qquad (15)$$

The exponential temperature profile

$$\theta = \exp[-(\alpha - \beta\dot{\xi})(\zeta - \xi)] \qquad (16)$$

satisfies boundary conditions (8–10) Combining Eqs (15) and (16), integrating, and simplifying results in the following second-order ordinary nonlinear differential equation for the position of the ablating surface ξ as a function of time:

$$\beta\ddot{\xi}[(\alpha - \beta\dot{\xi})\xi + 2] = (\alpha - \beta\dot{\xi})^2 \times \{(\alpha - \beta\dot{\xi})^2\xi - \dot{\xi}[(\alpha - \beta\dot{\xi})\xi + 1]\} \qquad (17)$$

It is of interest to note that the steady-state ablation rate checks with results published in the literature [7, 8] For large time, it is reasonable to assume that $(\alpha - \beta\dot{\xi})\xi \rightarrow \infty$ Hence, with $\ddot{\xi} \rightarrow 0$, Eq (17) reduces to

$$\dot{\xi}_{\infty} = \alpha/(\beta + 1) \qquad (18)$$

A short-time solution is readily found by expanding $\xi(\tau)$ in a Taylor's series about $\tau = 0$ and making use of initial conditions (11) and Eq (17) to obtain

$$\xi = 1 + (\alpha^4\tau^2)/[2\beta(\alpha + 2)] \qquad (19)$$

The results of a numerical step-by-step integration of Eq (17) are presented in Fig 2 for $\beta = 10$ and a range of α extending from 10 to 1000 The parameter β, which depends only on the physical properties of the solid, is applicable to nylon Physical values of time for $10^{-7} < \tau < 1$ correspond to $10^{-3} < t < 10{,}000$ sec

In any particular problem, the probable applicability of the solution can be gaged by calculating the premelt temperature distribution and comparing it with the exponential function assumed in Eq (16) When the value of α is high (high heat input to low conductivity materials), the assumed temperature distribution drops sharply and good agreement has been obtained with the calculated premelt distribution At $t = 0$, Eq (16) reduces to

$$\theta = \exp[-\alpha(\zeta - 1)]$$

and for α sufficiently high, most of the heat absorbed by the solid is contained in a "thermal" layer δ, which is small compared to the initial radius of the cylindrical cavity For $\delta/a \ll 1$, the effects of the nonplanar geometry are minimized and results[8] for a semi-infinite medium,

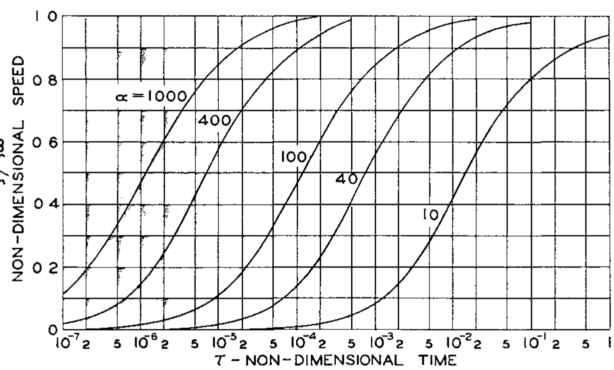

Fig 2 Ablation rate history: $\beta = 10$

ablating at its interface, will be approached in the limit For very low heat inputs $\alpha \rightarrow 0$, it is unlikely that the exponential temperature function assumed here is applicable However, many problems of practical interest will undoubtedly involve high thermal inputs

The results reported also should be applicable to finite geometries (tubes) up to the time corresponding to a rise in temperature of the unheated or outer surface

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Euler's Moment Equations for a Variable-Mass Unsymmetrical Top

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Nomenclature

B	=	body of variable mass
0	=	fixed point of the body
$0x_1x_2x_3$	=	body axes system
m	=	mass of a typical particle of the body
x_i	=	body-axes coordinates of mass particle m, $i = 1,2\ 3$
ω_i	=	angular velocity of the body B expressed in body axes system
d/dt	=	derivative with respect to any fixed (inertial) coordinate axes
$\delta/\delta t$	=	derivatives with respect to the body axes
ϵ_{ijk}	=	permutation tensor = { 1, if ijk is a cyclic permutation of 1, 2, 3; −1, if ijk is an anticyclic permutation of 1, 2, 3; 0 when any two of $i, j\ k$ are equal
V_i	=	dx_i/dt = absolute velocity of particle m, i e, velocity with respect to a fixed coordinate system
c_i	=	relative velocity of mass ejected by a particle m, i e, velocity relative to the body axes
u_i	=	$v_i + c_i$, absolute velocity of mass ejected by a particle m, i e, velocity relative to the fixed axes
$\dot{m}$	=	rate of the mass flow ejected by a particle m
F_i	=	external force acting on particle of mass m
f_i	=	$\dot{m}c_i$, reactive force acting on particle of mass m

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